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INVOLUTION CODIMENSIONS AND TRACE CODIMENSIONS OF MATRICES ARE ASYMPTOTICALLY EQUAL

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In memory of Shimshon Amitsur, our teacher and our friend

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ABSTRACT

We calculate the asymptotic growth of $t_n(M_p(F),*)$ and $c_n(M_p(F),*)$, the trace and ordinary *-codimensions of $p\times p$ matrices with involution. To do this we first calculate the asymptotic growth of t_n and then show that $c_n \simeq t_n$.

Introduction

Our main goal in this paper is to calculate the asymptotic behaviours of the multilinear *-codimensions of $p\times p$ matrices with or without trace, over a field of characteristic zero. The calculation will parallel the calculation of the asymptotic behaviour of the ordinary cocharacter of $p\times p$ matrices which was carried out in [12]: We use the theory of $*$-trace identities to calculate the asymptotic behaviour of the *-trace codimensions and then show that the $*$-trace codimensions and the $*$-codimensions are asymptotically equal.

The first step in this program is easily obtained by combining two known results. It follows from the work of Procesi and Loday–Procesi [8, 9, 10] that the multilinear trace codimensions of $p\times p$ matrices satisfy:

$$\text{(1)}\qquad \begin{aligned}&t_n(M_p, transpose) =\\ &\{d_\lambda | \lambda \in \text{Par}(2n), ht(\lambda)\le p, \text{ all rows of } \lambda \text{ are even}\}\end{aligned}$$

and

$$\text{(2)}\qquad \begin{aligned}&t_n(M_p, symplectic) =\\ &\sum\{d_\lambda | \lambda \in \text{Par}(2n), ht(\lambda)\le p, \text{ all columns of } \lambda \text{ are even}\}.\end{aligned}$$

The asymptotics of these two sums were calculated in [11]:

$$\text{(1)}\qquad \begin{aligned}&t_n(M_p, transpose) = U^{(1)}_{2,p}(2n) = S^{(1)}_p(2n) \simeq\\ &\left[\sqrt{p}^{p(p-1)/2}\cdot \frac{1}{p!}\Gamma\left(\frac{3}{2}\right)^{-p}\prod_{j=1}^{p}\Gamma\left(1+\frac{1}{2}j\right)\left(\frac{1}{2}\right)^{p-1}\right]\left(\frac{1}{\sqrt{2n}}\right)^{p(p-1)/2}\cdot p^{2n}.\end{aligned}$$

$$\text{(2)}\qquad \begin{aligned}&t_n(M_p, symplectic) = T^{(1)}_{2,p}(2n) = T^{(1)}_{2,2N}(2n)\\ &\simeq \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{N} 2^{(N^2+N+2)/4}\cdot N^{N(7N-1)/4}\cdot \frac{1}{N!}\prod_{j=1}^{N}\Gamma(2j+1)\right]\\ &\times\left(\frac{1}{2n}\right)^{N(2N+1)/2}(2N)^{2n}, \qquad \text{where } 2N = p.\end{aligned}$$

Hence, the remainder of this paper will be devoted to proving that $t_n(M_p,*) \simeq c_n(M_p,*)$ for each of transpose and symplectic involutions. The first two sections will be devoted to some general results pertaining to $*$-cocharacters. In section 1 we compare the S_n- and B_n-cocharacters and in section 2 we compare the B_n- and $\mathrm{GL}(a) \times \mathrm{GL}(b)$-cocharacters. In section 3 we specialize to the case of $p \times p$ matrices and draw various conclusions about the cocharacters from the existence of the conductor. Finally, in section 4 we prove our main result that $c_n(M_p,*) \simeq t_n(M_p,*)$ in all cases.

In this paper we chose to use B_n-cocharacters to study the relation between c_n and t_n. It seems probable that one could obtain our main result more quickly by using S_n-cocharacters instead, since these characters are better understood. But we feel that it is worthwhile to understand the B_n-cocharacter as well as the S_n-cocharacter and that our results on B_n-cocharacters are of independent interest.

1. Comparison of B_n- and S_n-cocharacters

The group B_n can be defined most succinctly as the wreath product of $Z/2Z$ with S_n. Here is a less succinct definition: B_n is the group of permutations of the set $X_n = \{x_1, x_1^*, \ldots, x_n, x_n^*\}$ generated by the symmetric group S_n acting via $\sigma(x_i) = x_{\sigma(i)}$ and $\sigma(x_i^*) = x_{\sigma(i)}^*$; and by the n transpositions which switch x_i and x_i^*. If we think of $*$ as an involution on X_n, then B_n is the set of permutations such that $\sigma(a^*) = \sigma(a)^*$ for all $a \in X_n$.

We next let V_n be the space of all polynomials in X_n which are homogeneous of degree n and are multilinear in the sense that $f(\alpha_1 x_1, \alpha_1 x_1^*, \ldots, \alpha_n x_n, \alpha_n x_n^*) = \alpha_1\alpha_2\cdots\alpha_n f(x_1, x_1^*, \ldots, x_n, x_n^*)$ for all $\alpha_1, \ldots, \alpha_n$ in the field F. For convenience, we will write $f(x_1, x_1^*, \ldots, x_n, x_n^*)$ as $f(x_1, \ldots, x_n)$. There is an obvious action of B_n on V_n given by $\sigma f(x_1, \ldots, x_n) = f(\sigma(x_1), \ldots, \sigma(x_n))$. This action is important from the point of view of $*$-polynomial identities. Let A be an F-algebra with involutions and let $I_n(A) =$ the $*$-polynomial identities of A contained in V_n. Then $I_n(A)$ is a B_n-submodule of V_n and so we may study $\chi_{B_n}(A)$, the nth multilinear $*$- cocharacters of A, defined to be the B_n-character of the quotient $V_n/I_n(A)$. By restriction, $V_n/I_n(A)$ is also a module for the symmetric group and we will denote its S_n-character by $\chi_{S_n}(A)$.

If A is an algebra with both involution and trace, then we may extend the above definitions to include both operations: $\bar{V}_n$ will be the vector space of all degree n, multilinear, mixed trace polynomials in X_n. So, for example, $\bar{V}_2$ will be spanned by V_2 together with

$$\{x_1 \operatorname{tr}(x_2), x_1^* \operatorname{tr}(x_2), x_1 \operatorname{tr}(x_2^*), x_1^* \operatorname{tr}(x_2^*), x_2 \operatorname{tr}(x_1), \ldots,$$
$$x_2^* \operatorname{tr}(x_1^*), \operatorname{tr}(x_1x_2), \operatorname{tr}(x_1^*x_2), \operatorname{tr}(x_1x_2^*), \operatorname{tr}(x_1^*x_2^*), \operatorname{tr}(x_1)\operatorname{tr}(x_2), \ldots, \operatorname{tr}(x_1^*)\operatorname{tr}(x_2^*)\}.$$

Then, $\bar{I}_n(A)$ will be the $*$-trace polynomial identities for A contained in $\bar{V}_n$; it will be a B_n-submodule; and we may use the quotient $\bar{V}_n/\bar{I}_n(A)$ to define the characters $\bar{\chi}_{B_n}(A)$ and $\bar{\chi}_{S_n}(A)$.

Asides: (1) In the cases we will study here, the algebra A will satisfy the relation $\operatorname{tr}(a^*) = \operatorname{tr}(a)$, for all $a \in A$. One could include the relation $\operatorname{tr}(u^*) = \operatorname{tr}(u)$, for all monomials u, in the definition of $\bar{V}_n$. At any rate, the quotient $\bar{V}_n/\bar{I}_n(A)$ will be the same in both cases since, if the relations $\operatorname{tr}(u^*) = \operatorname{tr}(u)$ are not already built into $\bar{V}_n$, they will be included in $\bar{I}_n(A)$.

(2) The only cocharacters we study in this paper are $*$-cocharacters with or without trace. As an aside, we compare this theory to the more familiar theory of ordinary cocharacters. In that case, the space of multilinear polynomials is identified with FS_n, and this identification is an S_n-isomorphism, taking the S_n-action on FS_n to be left multiplication. In the case of multilinear $*$-polynomials there is a similar B_n-isomorphism with the regular representation FB_n. Specially, $\sigma \in B_n$ may be identified with the $*$-monomial $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_n)$.

For trace polynomials without $*$, there is an identification of pure trace polynomials with FS_n. To make the identification an S_n-isomorphism, one uses the conjugation action of S_n on FS_n. If the trace is non-degenerate, one may also identify mixed trace polynomials with elements of FS_{n+1} under the conjugation action of FS_n. In the case of polynomials with $*$ and trace, Loday and Procesi have a B_n-isomorphism between $F[S_{2n}/B_n]$ and the space of pure trace $*$-polynomials, modulo the relation of the previous remark

(3) It follows from (2) that $\dim V_n = 2^n n!$ and that the space of degree n pure $*$-trace polynomials is $(2n)!/2^n n! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ under the relation $\operatorname{tr}(a^*) = \operatorname{tr}(a)$. By Sterling's formula, this is asymptotic to $\sqrt{\frac{1}{\pi}\frac{1}{\sqrt{n}}2^n n!}$. At any rate, it is close to $2^n n!$ Using the standard trick of identifying a mixed trace polynomial $f \in \bar{V}_n$ with the pure trace polynomial $\operatorname{tr}(x_{n+1}f) \in \bar{V}_{n+1}$, it follows that $\dim \bar{V}_n = (2(n+1)!)/2^{n+1}(n+1)!$ which, by the above remark, is nearly $2^{n+1}(n+1)!$

Irreducible S_n-characters are denoted $\chi_\lambda, \lambda \in \mathrm{Par}(n)$ and irreducible B_n-characters are denoted $\chi_{\mu,\nu}, \mu \in \mathrm{Par}(a), \nu \in \mathrm{Par}(b), a+b=n$. Hence we may write

$$\chi_{B_n}(A) = \sum_{a+b=n} \sum_{\substack{\mu\in\mathrm{Par}(a)\\ \nu\in\mathrm{Par}(b)}} m_{\mu,\nu}\chi_{\mu,\nu},$$

$$\chi_{S_n}(A) = \sum_{\lambda\in\mathrm{Par}(n)} m_\lambda \chi_\lambda,$$

$$\bar\chi_{B_n}(A) = \sum_{a+b=n} \sum_{\substack{\mu\in\mathrm{Par}(a)\\ \nu\in\mathrm{Par}(b)}} \bar m_{\mu,\nu}\chi_{\mu,\nu}$$

and

$$\bar\chi_{S_n}(A) = \sum_{\lambda\in\mathrm{Par}(n)} \bar m_\lambda \chi_\lambda.$$

If the B_n-character $\chi_{\mu,\nu}$ is induced down to S_n, the resulting character is the outer tensor product $\chi_\mu\hat\otimes\chi_\nu$. We mentioned this result to a number of experts who were unanimous in telling us that it was easy but that they knew of no reference. We include a proof in the appendix of this fact and of the rule for calculating $\chi_{\mu_1,\nu_1}\hat\otimes\chi_{\mu_2,\nu_2}$. The outer tensor product $\chi_\mu\hat\otimes\chi_\nu$ can be calculated by the Littlewood–Richardson rule. cf. [7] chapter 16. If we write the coefficients as $\chi_\mu\hat\otimes\chi_\nu = \sum_{\lambda\in\mathrm{Par}(n)} c(\lambda;\mu,\nu)\chi_\lambda$, our first lemma is now immediate:

LEMMA 1:
(a) $m_\lambda = \sum_{\mu,\nu} c(\lambda;\mu,\nu) m_{\mu,\nu}$.
(b) $\bar m_\lambda = \sum_{\mu,\nu} c(\lambda;\mu,\nu)\bar m_{\mu,\nu}$, *for all* $\lambda \in \mathrm{Par}(n)$.

We close this section with a comparison of the trace and non-trace cocharacters.

LEMMA 2:
(a) $m_\lambda \le \bar m_\lambda$, *for all* $\lambda \in \mathrm{Par}(n)$.
(b) $m_{\mu,\nu} \le \bar m_{\mu,\nu}$, *for all* $\mu \in \mathrm{Par}(a), \nu \in \mathrm{Par}(b), a+b=n$.

Proof: The inclusion map $V_n \to \bar V_n$ preserves the B_n-action and takes $I_n(A)$ into $\bar I_n(A)$. Hence, there is a B_n-injection (and *a fortiori* an S_n-injection) $V_n/I_n(A) \to \bar V_n/\bar I_n(A)$.

2. Poincaré series and cocharacters

Our first task in this section is to extend Giambruno's results from [4] in two directions. Firstly, that work studies algebras with involution and we will need the corresponding results for algebras with both involution and trace. Secondly, [4] compares the B_n-cocharacters with $\mathrm{GL}(n) \times \mathrm{GL}(n)$-cocharacters. We will need to compare it to $\mathrm{GL}(a) \times \mathrm{GL}(b)$-cocharacters. We will denote the irreducible $\mathrm{GL}(n)$ character corresponding to the partition λ by $\varphi_\lambda^{(n)}$.

For the first task, given any $\mathrm{GL}(n) \times \mathrm{GL}(n)$-module M, let M^{mult} be the space of all $m \in M$ such that $dm = \alpha_1 \cdots \alpha_n m$, where

$$d = (\mathrm{diag}(\alpha_1, \ldots, \alpha_m), \mathrm{diag}(\alpha_1, \ldots, \alpha_m)) \in \mathrm{GL}(n) \times \mathrm{GL}(n).$$

The group B_n has an embedding into $\mathrm{GL}(n) \times \mathrm{GL}(n) \equiv \mathrm{GL}(U) \times \mathrm{GL}(V)$ which makes the following true.

THEOREM (Giambruno): *Let $M \subseteq (U \oplus V)^{\otimes n}$ be a $\mathrm{GL}(U) \times \mathrm{GL}(V)$-module with character $\sum_{|\mu|+|\nu|=n} a_{\mu,\nu}\varphi_\mu^{(n)} \otimes \varphi_\nu^{(n)}$. Then M^{mult} has B_n-character $\sum_{|\mu|+|\nu|=n} a_{\mu,\nu}\chi_{\mu,\nu}$.*

COROLLARY 3: *Let M be any finite-dimensional, polynomial $\mathrm{GL}(U) \times \mathrm{GL}(V)$-module with character $\sum_{|\mu|+|\nu|=n} a_{\mu,\nu}\varphi_\mu^{(n)} \otimes \varphi_\nu^{(n)}$. Then M^{mult} is a B_n-module with character $\sum_{|\mu|+|\nu|=n} a_{\mu,\nu}\chi_{\mu,\nu}$.*

Proof: M is a homomorphic image of a direct sum of copies of $(U \oplus V)^{\otimes n}$.

We remark for future reference that the generic diagonal matrix $(\mathrm{diag}(t_1, \ldots, t_n), \mathrm{diag}(u_1, \ldots, u_n)) \in \mathrm{GL}(n) \times \mathrm{GL}(n)$ has trace $\sum_{|\mu|+|\nu|=n} a_{\mu,\nu}$ $S_\mu(t_1, \ldots, t_n)S_\nu(u_1, \ldots, u_n)$ on M, where S_μ and S_ν are Schur functions.

Now, fix an algebra A with a trace and an involution. For $a, b \geq 0$ let $\bar{U}(a,b)$ be the generic algebra (with trace and involution) for A generated by a symmetric elements $s_1, s_2, \ldots, s_a$ and b skew symmetric elements $k_1, k_2, \ldots, k_b$. Note that each $\bar{U}(a,b)$ is an algebra with trace and involution; each $\bar{U}(a,b)$ has a grading by total degree, $\bar{U}(a,b) = \sum_{n=0}^{\infty} \bigoplus \bar{U}_n(a,b)$; each $\bar{U}_n(a,b)$ is a module for $\mathrm{GL}(a) \times \mathrm{GL}(b)$ in a natural fashion; and each $\bar{U}(a,b)$ has an $(a+b)$-fold grading which defines a Poincaré series $\bar{f}_{a,b}(t_1, \ldots, t_1, u_1, \ldots, u_b)$, symmetric in the t's and in the u's.

For each $n, \bar{U}_n(n,n)$ has multilinear part equal to $\bar{V}_n/\bar{I}_n(A)$, defined in section 1. It follows from Corollary 3 that $\bar{U}_n(n,n)$ has $\mathrm{GL}(n) \times \mathrm{GL}(n)$-character $\sum_{|\mu|+|\nu|=n} \bar{m}_{\mu,\nu}\varphi_\mu^{(n)} \otimes \varphi_\nu^{(n)}$. Our next lemma generalizes this fact.

LEMMA 4: *For any a, b,*

$$\bar{f}_{a,b}(t_1, \ldots, t_a, u_1, \ldots, u_b) = \sum_{n=0}^{\infty} \sum_{|\mu|+|\nu|=n} \bar{m}_{\mu,\nu} S_\mu(t_1, \ldots, t_a) S_\nu(u_1, \ldots, u_b).$$

Proof: If $A \geq a$ and $B \geq b$ there is a multidegree preserving injection $\bar{U}(a,b) \to \bar{U}(A,B)$. It follows that $\bar{f}_{a,b} = \bar{f}_{A,B}|_{\substack{t_{a+1} = \cdots = t_A = 0 \\ u_{b+1} = \cdots = u_B = 0}}$. Hence, if $\bar{U}(A,B)$ has Poincaré series $\bar{f}_{A,B} = \sum_{n=0}^{\infty} \sum_{|\mu|+|\nu|=n} C_{\mu,\nu} S_\mu(t_1, \ldots, t_A) S_\nu(u_1, \ldots, u_B)$, then $\bar{U}(a,b)$ has Poincaré series

$$\bar{f}_{a,b} = \sum_{n=0}^{\infty} \sum_{|\mu|+|\nu|=n} C_{\mu,\nu} S_\mu(t_1, \ldots, t_a) S_\nu(u_1, \ldots, u_b),$$

with the usual convention that $S_\mu(t_1, \ldots, t_a) S_\nu(u_1, \ldots, u_b) = 0$ if $ht(\mu) > a$ or $ht(\nu) > b$.

In the special case of $a = b = n$, it follows from the previous remark that $C_{\mu,\nu} = \bar{m}_{\mu,\nu}$ for $|\mu| + |\nu| = n$. In this case, of course, $ht(\mu), ht(\nu) \leq n$ and the lemma now follows.

The same arguments hold equally well if we consider algebras with $*$ only. Let $U(a,b)$ be the generic algebra for A, as an algebra with involution, on a symmetric generators and b skew symmetric generators. If $U(a,b)$ has Poincaré series $f_{a,b}(t_1, \ldots, t_a, u_1, \ldots, u_b)$, then as above

LEMMA 5: *For any a, b,*

$$f_{a,b}(t_1, \ldots, t_a, u_1, \ldots, u_b) = \sum_{n=0}^{\infty} \sum_{|\mu|+|\nu|=n} m_{\mu,\nu} S_\mu(t_1, \ldots, t_a) S_\nu(u_1, \ldots, u_b).$$

As a consequence of Lemma 5 we get a polynomial bound for the $m_{\mu,\nu}$ if we restrict μ, ν to a strip.

THEOREM 6: *Let A be an algebra with involution which satisfies some $*$-identity and with B_n-cocharacter $\chi_{B_n}(A) = \sum_{|\mu|+|\nu|=n} m_{\mu,\nu} \chi_{\mu,\nu}$. For a fixed k, let*

$$f(n) = \sum \{m_{\mu,\nu} | ht(\mu), ht(\nu) \leq k, \ |\mu| + |\nu| = n\}.$$

Then $f(n)$ is bounded by a polynomial in n.

Proof: Let $U = U(k,k)$. By the preceding lemma U has Poincaré series $\sum_{n=0}^{\infty} \sum m_{\mu,\nu} S_\mu(t) S_\nu(u)$ where the second summation is over $|\mu| + |\nu| = n$,

$ht(\mu) \le k, ht(\nu) \le k$. Now decompose U by total degree $U = \sum \bigoplus U_n$ and let $g(n) = \dim U_n$. Hence,

$$g(n) = \sum\nolimits_{|\mu|+|\nu|=n} m_{\mu,\nu} S_\mu(1,\ldots,1) S_\nu(1,1,\ldots,1) = \sum\nolimits_{|\mu|+|\nu|=n} m_{\mu,\nu} d_\mu^{(k)} d_\nu^{(k)},$$

where by $d_\mu^{(k)}, d_\nu^{(k)}$ we mean the degrees of the GL(k)-modules on μ and ν. Since these degrees are at least one for $ht(\mu), ht(\nu) \le k, g(n) \ge f(n)$.

By Amitsur's Theorem ([1], see also [6]), since U satisfies a $*$-identity it must also satisfy an ordinary polynomial identity. Hence, by Berele's Theorem ([2]) U has finite GK dimension. But, $g(n)$ is a growth function for U, so $g(n)$ is bounded by a polynomial in n and so $f(n)$ must also be bounded by a polynomial in n.

COROLLARY 7: *Let $A = p \times p$ matrices with either symplectic or transpose involution. Then $f(n) = \sum_{|\mu|+|\nu|=n} m_{\mu,\nu}$ is polynomially bounded.*

Proof: By [5] there are constants ℓ_1 and ℓ_2 such that $m_{\mu,\nu} = 0$ unless $ht(\mu) \le \ell_1$ and $ht(\nu) \le \ell_2$.

We will show in the next section that $\bar{m}_{\mu,\nu} = 0$ unless $ht(\mu) \le a$ and $ht(\nu) \le b$, where $a + b = p^2$. Since $\ell_1 + \ell_2$ also equals p^2 it follows that $(a, b) = (\ell_1, \ell_2)$. In the orthogonal case $(\ell_1, \ell_2) = (\frac{1}{2}p(p+1), \frac{1}{2}p(p-1))$ and in the symplectic case $p = 2N$ and $(\ell_1, \ell_2) = (2N^2 - N, 2N^2 + N)$.

3. Facts about multiplicities

For the rest of this paper we will specialize to the case of $A = p \times p$ matrices with either symplectic or transpose involution. We will continue to use the notation $m_\lambda, \bar{m}_\lambda, m_{\mu,\nu}$ and $\bar{m}_{\mu,\nu}$ to denote the multiplicities of the irreducible components in these particular A's. For the reader's convenience we record some facts from [3]:

THEOREM 8:

(a) (= Theorem 11.2 of [3]) *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p^2})$ with $\lambda_{p^2} \ge 2$, then $m_\lambda = \bar{m}_\lambda$.*

(b) (= Lemma 11.1 of [3]) *If $\kappa = (1^{p^2})$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p^2})$, let $\lambda + \kappa$ denote $(\lambda_1 + 1, \ldots, \lambda_{p^2} + 1)$. Then $\bar{m}_\lambda = \bar{m}_{\lambda+\kappa}$*

(c) (follows from Theorems 5.3 and 9.6 of [3]) *If $ht(\lambda) > p^2$, then $\bar{m}_\lambda = 0$.*

As an immediate consequence we get

THEOREM 9: *Let $\mu = (\mu_1, \ldots, \mu_a), \nu = (\nu_1, \ldots, \nu_b)$ be such that $\mu_a, \nu_b \geq 2$ and $a + b = p^2$. Then $m_{\mu,\nu} = \bar{m}_{\mu,\nu}$.*

Proof: By the Littlewood–Richardson rule, there is a partition $\lambda = (\lambda_1, \ldots, \lambda_{p^2})$ with $\lambda_{p^2} \geq 2$ and $c(\lambda; \mu, \nu) \neq 0$. By Theorem 8(a) $\bar{m}_\lambda$ and m_λ are equal. Now apply Lemma 1:

$$\begin{aligned} 0 =& \bar{m}_\lambda - m_\lambda \\ =& \sum_{\alpha,\beta} c(\lambda, \alpha, \beta)\bar{m}_{\alpha,\beta} - \sum_{\alpha,\beta} c(\lambda, \alpha, \beta) m_{\alpha,\beta} \\ =& \sum_{\alpha,\beta} c(\lambda, \alpha, \beta)(\bar{m}_{\alpha,\beta} - m_{\alpha,\beta}). \end{aligned}$$

But, by Lemma 2(b), $\bar{m}_{\alpha,\beta} - m_{\alpha,\beta}$ is always non-negative, and the Littlewood–Richardson coefficients $c(\lambda, \alpha, \beta)$ are always non-negative. Hence $c(\lambda; \mu, \nu) \neq 0$ implies that $\bar{m}_{\mu,\nu} = m_{\mu,\nu}$.

Our next goal is to get an analogue of 8(b) and 8(c) for $\bar{m}_{\mu,\nu}$.

LEMMA 10:

(a) *If $ht(\mu) + ht(\nu) > p^2$ then $\bar{m}_{\mu,\nu} = 0$.*

(b) *There exists unique $a + b = p^2$ such that $\bar{m}_{(1^a,1^b)} = 1$. For all other $c + d = p^2$, $\bar{m}_{(1^c,1^d)} = 0$.*

Proof: (a) In this case $c(\lambda; , \mu, \nu) \neq 0$ for some λ with $ht(\lambda) > p^2$. So, if $\bar{m}_{\mu,\nu} \neq 0$ then $\bar{m}_\lambda \neq 0$ by Lemma 2(b) contradicting Theorem 8(c).

(b) First, $\bar{m}_{(1^{p^2})} = 1$ as a special case of Theorem 8(b). Hence $1 = \sum_{\mu,\nu} c(1^{p^2}; \mu, \nu)\bar{m}_{\mu,\nu}$. But $c(1^{p^2}; \mu, \nu) \neq 0$ only if $\mu = 1^a$ and $\nu = 1^b$ for some $a + b = p^2$, in which case $c(1^{p^2}; \mu, \nu) = 1$. The lemma follows.

More is true: The polynomial corresponding to $\chi_{(1^{p^2})}$ and hence to $\chi_{(1^a,1^b)}$ is a pure trace polynomial. So it is central. In particular it is not a zero divisor, since the ring of generic matrices with trace is prime. Translating to characters gives the following lemma:

LEMMA 11:

(a) *If $ht(\mu) > a$ or $ht(\nu) > b$ then $\bar{m}_{\mu,\nu} = 0$.*

(b) *Let $ht(\mu) \leq a$, $ht(\nu) \leq b$ and denote $(\mu_1+1, \ldots, \mu_a+1)$, $(\nu_1+1, \ldots, \nu_b+1)$ as $\mu + (1^a)$ and $\nu + (1^b)$, respectively. Then $\bar{m}_{\mu+1^a,\nu+1^b} \geq \bar{m}_{\mu,\nu}$.*

Proof: (a) Let $f(x_1, \ldots, x_{p^2}) \in \bar{V}_{p^2}$ be a non-identity, central polynomial for A which generates a B_{p^2}-submodule with character $\chi_{(1^a,1^b)}$; and let $g(x_1, \ldots, x_n) \in$

$\bar{V}_n$ be a non-identity which generates a B_{p^2}-submodule with character $\chi_{\mu,\nu}$. Then $f(x_1,\ldots,x_{p^2})g(x_{p^2+1},\ldots,x_{p^2+n})$ is a non-identity for A and is contained in a B_{n+p^2}-submodule of $\chi_{1^a,1^b}\hat{\otimes}\chi_{\mu,\nu}$. Hence, $\bar{m}_{\alpha,\beta}\neq 0$ for some $\chi_{\alpha,\beta}$ occurring in this product. By Theorem A.1 from the appendix, $\chi_{1^a,1^b}\hat{\otimes}\chi_{\mu,\nu}$ is a sum of terms of the form $\chi_{\alpha,\beta}$ where α occurs in $\chi_{(1^a)}\hat{\otimes}\chi_\mu$ and β occurs in $\chi_{(1^b)}\hat{\otimes}\chi_\nu$. By Littlewood–Richardson, $ht(\alpha)\geq\max\{a,ht(\mu)\}$ and $ht(\beta)\geq\max\{b,ht(\nu)\}$. So, if $ht(\mu)>a$ or $ht(\nu)>b$, $ht(\alpha)+ht(\beta)$ would always be greater than $a+b=p^2$. But, this is impossible by Lemma 10(a).

(b) Let $f\in\bar{U}(a,b)$ correspond to the $\mathrm{GL}(a)\times\mathrm{GL}(b)$-character $\varphi_{(1^a)}\otimes\varphi_{(1^b)}$. Multiplication by f gives a $\mathrm{GL}(a)\times\mathrm{GL}(b)$ injection from $\bar{U}(a,b)$ to itself and the image of $\bar{U}_n(a,b)$ has character $(\varphi_{(1^a)}\otimes\varphi_{(1^b)})$ times the character of $\bar{U}_n(a,b)$. The lemma now follows from the Littlewood–Richardson rule.

THEOREM 12: *For all* μ,ν, $\bar{m}_{\mu+1^a,\nu+1^b}=\bar{m}_{\mu,\nu}$.

Proof: Given μ,ν, choose any λ with $c(\lambda;\mu,\nu)\neq 0$. Then $\bar{m}_\lambda=\bar{m}_{\lambda+(1^{p^2})}$ by Theorem 8(b). Hence, by Lemma 1(b), $\bar{m}_\lambda=\sum_{\alpha,\beta}c(\lambda+(1^{p^2});\alpha,\beta)\bar{m}_{\alpha,\beta}$. But, $c(\lambda+(1^{p^2});\alpha,\beta)\neq 0$ only if $ht(\alpha)+ht(\beta)\geq p^2$ and $\bar{m}_{\alpha,\beta}\neq 0$ only if $ht(\alpha)\leq a$, $ht(\beta)\leq b=p^2-a$. Hence, we may restrict the sum to those α,β with $ht(\alpha)=a$ and $ht(\beta)=b$. So, we may replace α by $\alpha+(1^a)$ and β by $\beta+(1^b)$. So

$$\bar{m}_\lambda=\sum_{\alpha,\beta}c(\lambda+(1^{p^2});\alpha+(1^a),\beta+(1^b))\bar{m}_{\alpha+(1^a),\beta+(1^b)}.$$

It is an exercise in the Littlewood–Richardson rule to prove that $c(\lambda+(1^{p^2});\alpha+(1^a),\beta+(1^b))=c(\lambda;\alpha,\beta)$. So,

$$\bar{m}_\lambda=\sum_{\alpha,\beta}c(\lambda;\alpha,\beta)\bar{m}_{\alpha+(1^a),\beta+(1^b)}.$$

On the other hand, by Lemma 1(b) again,

$$\bar{m}_\lambda=\sum_{\alpha,\beta}c(\lambda;\alpha,\beta)\bar{m}_{\alpha,\beta}.$$

The Theorem now follows from Lemma 11(b).

4. Proof of Main Theorem

LEMMA 13: *If $|\mu|+|\nu|=n$, then $\bar{m}_{\mu,\nu} \le f(n)$, for some polynomial f depending on A.*

Proof: Let $\mu' = \mu + (2^a) = (\mu_1+2, \mu_2+2, \ldots, \mu_a+2)$ and $\nu' = \nu + (2^b) = (\nu_1+2, \ldots, \nu_b+2)$. By Theorem 12, $\bar{m}_{\mu,\nu} = \bar{m}_{\mu',\nu'}$, and by Theorem 9, $\bar{m}_{\mu',\nu'} = m_{\mu',\nu'}$. Finally, by Corollary 7, $m_{\mu',\nu'}$ is polynomially bounded, say by $g(n)$. Thus $\bar{m}_{\mu,\nu} \le g(|\mu'|+|\nu'|) = g(n+2p^2)$ and the lemma follows.

LEMMA 14: *Let $ht(\mu), ht(\nu)$ be bounded by constants a and b. Then the Littlewood–Richardson coefficients $c(\lambda;\mu,\nu)$ are bounded by a polynomial in $|\mu|+|\nu|$.*

Proof: Write $\mu = (\mu_1, \ldots, \mu_a)$. Then $\langle \chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_a)}, \chi_\mu \rangle \ne 0$ and so it suffices to prove that the multiplicities in $\chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_a)} \hat{\otimes} \chi_\nu$ are polynomial in $|\mu|+|\nu| = n$.

We may assume by induction that $\chi_{(\mu_2)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_a)} \hat{\otimes} \chi_\nu = \sum a(\lambda)\chi_\lambda$ where $a(\lambda) \le (n-\mu_1)^k$ for some k and where λ has height bounded by $ht(\lambda) \le a+b-1$. Let $q = \chi_{(\mu_1)} \hat{\otimes} \cdots \hat{\otimes} \chi_{(\mu_a)} \hat{\otimes} \chi_\nu = \sum a(\lambda)(\chi_{(\mu_1)} \hat{\otimes} \chi_\lambda)$. By Young's rule $\chi_{(\mu_1)} \hat{\otimes} \chi_\lambda = \sum \chi_\eta$ where $|\eta| = n, \lambda \subseteq \eta$ and where η/λ is a (skew) horizontal strip of size μ_1. Thus

$$q = \sum_\eta b(\eta)\chi_\eta$$

and

$$\begin{aligned} b(\eta) &= \sum \{a(\lambda) | \lambda \subseteq \eta, \eta/\lambda \text{ horizontal}\} \\ &\le (n-\mu_1)^k \# \{\lambda | \lambda \subseteq \eta, \eta/\lambda \text{ horizontal}\}. \end{aligned}$$

For a given η the cardinality of this set is $\le \Pi_{i=1}^{a+b-1}(\eta_i - \eta_{i+1} + 1) \le n^{a+b-1}$. Thus, $b(\eta) \le (n-\mu_1)^k n^{a+b-1} \le n^{k+a+b-1}$.

Before proceeding with the proof, we fix a few notations:

$$\Lambda_\ell(n) = \{\lambda \in \mathrm{Par}(n) | ht(\lambda) \le \ell\}, \quad d_{\mu,\nu} = \deg \chi_{\mu,\nu} \quad \text{and} \quad d_\lambda = \deg \chi_\lambda.$$

LEMMA 15: *Let $\mu \in \Lambda_{\ell_1}(n_1)$ and $\nu \in \Lambda_{\ell_2}(n_2)$ be such that $\mu_{\ell_1} \le C$ for some constant C. Then $c(\lambda;\mu,\nu) \ne 0$ only if $\lambda \in \Lambda_{\ell_1+\ell_2}(n_1+n_2)$ and $\lambda_{\ell_1+\ell_2} \le C$.*

Proof: Follows from the Littlewood–Richardson rule.

LEMMA 16: *Let*

$$q = \sum_{\substack{|\mu|+|\nu|=n \\ \mu_a \le 1}} \bar{m}_{\mu,\nu} d_{\mu,\nu}.$$

Then there are constants c and g such that $q \le c \cdot n^g (a+b-1)^n$.

Proof: Since $\chi_{\mu,\nu}$ restricts to $\chi_\mu \hat{\otimes} \chi_\nu$,

$$d_{\mu,\nu} = \sum\nolimits_{\lambda \in \Lambda_{a+b}(n)} c(\lambda; \mu, \nu) d_\lambda \le n^d \sum_{\substack{\lambda \in \Lambda_{a+b}(n) \\ \lambda_{a+b} \le 1}} d_\lambda,$$

by Lemma 15. But $\bar{m}_{\mu,\nu} \le n^r$ by Lemma 13, so

$$q \le n^{r+d} \sum_{\substack{\lambda \in \Lambda_{a+b}(n) \\ \lambda_{a+b} \le 1}} d_\lambda \le c \cdot n^g (a+b-1)^n$$

by [11], proof of Theorem 1.

THEOREM: *Let A be $p \times p$ matrices with involution. Then the trace $*$-cocharacters $t_n(A)$ and the $*$-cocharacter $c_n(A)$ are asymptotically equal.*

Proof: $t_n(A) = \sum_{|\mu|+|\nu|=n} \bar{m}_{\mu,\nu} d_{\mu,\nu}$ and $c_n(A) = \sum_{|\mu|+|\nu|=n} m_{\mu,\nu} d_{\mu,\nu}$, hence $t_n(A) - c_n(A) = \sum_{|\mu|+|\nu|=n} (\bar{m}_{\mu,\nu} - m_{\mu,\nu}) d_{\mu,\nu}$. The difference $\bar{m}_{\mu,\nu} - m_{\mu,\nu}$ is zero unless $\mu_a < 2$ or $\nu_b < 2$ by Theorem 9. Hence

$$t_n(A) - c_n(A) \le \sum_{\substack{\mu_a \le 1 \\ |\mu|+|\nu|=n}} \bar{m}_{\mu,\nu} d_{\mu,\nu} + \sum_{\substack{\nu_b \le 1 \\ |\mu|+|\nu|=n}} \bar{m}_{\mu,\nu} d_{\mu,\nu}.$$

By Corollary 7, $\bar{m}_{\mu,\nu}$ is polynomially bounded and so $t_n(A) - c_n(A) \le$ a polynomial times $(a+b-1)^n$ by Lemma 16. Since $t_n(A)$ has exponential behavior $p^{2n} = (a+b)^n, c_n \simeq t_n$.

Appendix: Two theorems on B_n-characters

In this appendix we prove two theorems on B_n-characters which seem to be well known but which we have not been able to find in the literature. As in the rest of this paper we denote irreducible B_n-characters by $\chi_{\mu,\nu}, |\mu| + |\nu| = n$ and irreducible S_n-characters by $\chi_\lambda, |\lambda| = n$. Also we use $c(\lambda; \mu, \nu)$ to denote the Littlewood–Richardson coefficients $\chi_\mu \otimes \chi_\nu = \sum_\lambda c(\lambda; \mu, \nu) \chi_\lambda$. Here, now, are our two theorems:

THEOREM A.1: *$\chi_{\mu_1,\nu_1}\hat{\otimes}\chi_{\mu_2,\nu_2} = \sum_{\lambda_1,\lambda_2} c(\lambda_1;\mu_1,\mu_2)\cdot c(\lambda_2;\nu_1,\nu_2)\chi_{\lambda_1,\lambda_2}$. Or, more succinctly, $\chi_{\mu_1,\nu_1}\hat{\otimes}\chi_{\mu_2,\nu_2} = \chi_{\mu_1\hat{\otimes}\mu_2,\nu_1\hat{\otimes}\nu_2}$.*

THEOREM A.2: *Let $|\mu|+|\nu| = n$. Then $\chi_{\mu,\nu}\downarrow S_n = \chi_\mu\hat{\otimes}\chi_\nu$.*

Theorem A.2 follows easily from Theorem A.1: $\chi_{\mu,\nu} = \chi_{\mu,(0)}\hat{\otimes}\chi_{(0),\nu}$, where (0) denotes the trivial partition. And $\chi_{\mu,(0)}$ restricts to χ_μ and $\chi_{(0),\nu}$ restricts to χ_ν. So, the rest of the appendix will be devoted to the proof of Theorem A.1.

We now adopt the notations of the appendix of [5]. In particular, $f_1 = \frac{1}{2}(1+g)$, $f_2 = \frac{1}{2}(1-g)$ and $J_{\mu_i,\nu_i} = (A\sim S_{n_i})[(T^{\ell_i}(f_1)\otimes T^{m_i}(f_2))\otimes(e_{\mu_i}\otimes e_{\nu_i})]$, $i=1,2$, corresponds to χ_{μ_i,ν_i}, where $\ell_i = |\mu_i|, m_i = |\nu_i|$. Hence $\chi_{\mu_1,\nu_1}\hat{\otimes}\chi_{\mu_2,\nu_2}$ is the $A\sim S_n$-character of the $A\sim S_n$ left ideal

$$\begin{aligned}J =&(A\sim S_n)[(A\sim S_{n_1})e_{\mu_1,\nu_1}\otimes(A\sim S_{n_2})e_{\mu_2,\nu_2}]\\ =&(A\sim S_n)((A\sim S_{n_1})\otimes(A\sim S_{n_2}))(e_{\mu_1,\nu_1}\otimes e_{\mu_2,\nu_2})\\ =&(A\sim S_n)((T^{\ell_1}(f_1)\otimes T^{m_1}(f_2)\otimes T^{\ell_2}(f_1)\otimes T^{m_2}(f_2))\\ &\otimes(e_{\mu_1}\otimes e_{\nu_1}\otimes e_{\mu_2}\otimes e_{\nu_2}).\end{aligned}$$

Let

$$J_2 = (A\sim S_n)((T^{\ell_1+\ell_2}(f_1)\otimes T^{m_1+m_2}(f_2))\otimes(e_{\mu_1}\otimes e_{\mu_2}\otimes e_{\nu_1}\otimes e_{\nu_2}))..$$

Clearly, the $A\sim S_n$ character that corresponds to J_2 is $\chi_{\mu_1,\mu_2}\hat{\otimes}\chi_{\nu_1,\nu_2}$.

We next show that there is an algebra automorphism $\psi: A\sim S_n\to A\sim S_n$ with $\psi(J_1) = J_2$. This will imply that $J_1\cong J_2$ as left $A\sim S_n$ modules which will prove the theorem.

Assume W.L.O.G. that $\ell_2\le m_1$.

Let $\theta\in S_n$ be given by $\theta =$

$$\begin{pmatrix}1,\ldots,\ell_1, & \ell_1+1,\ldots,\ell_1+\ell_2, & \ell_1+\ell_2+1,\ldots,\ell_1+\ell_2+m_1, & \ell_1+\ell_2+m_1+1,\ldots,n\\ 1,\ldots,\ell_1, & \ell_1+m_1+1,\ldots,\ell_1+m_1+\ell_2, & \ell_1+1,\ldots,\ell_1+m_1, & \ell_1+\ell_2+m_1+1,\ldots,n\end{pmatrix}$$

Let $S_{\ell_1}\times S_{m_1}\times S_{\ell_2}\times S_{m_2}\subseteq S_n$ be the Young subgroup $S_{\ell_1} = S\{1,\ldots,\ell_1\}, S_{m_1} = S\{\ell_1+1,\ldots,\ell_1+m_1\}$, etc. For $\eta\in S_{\ell_1}\times S_{m_1}\times S_{\ell_2}\times S_{m_2}$ write $\eta = (\pi_1,\pi_2,\pi_2,\pi_4)$ and define π_2' and π_3' via: $\pi_2'\in S_{m_1} = S\{\ell_1+\ell_2+1,\ldots,\ell_1+\ell_2+m\}, \pi_2'(i) = \pi_2(i-\ell_2)+\ell_2$ and $\pi_3'\in S_{\ell_2} = S\{\ell_1+1,\ldots,\ell_1+\ell_2\}, \pi_3'(j) = \pi_3(j+m_1)-m_1$. It is easy to check that $\theta^{-1}\eta\theta = \theta^{-1}(\pi_1,\pi_2,\pi_3,\pi_4)\theta = (\pi_1,\pi_3',\pi_2',\pi_4)$, so $\theta^{-1}(e_{\mu_1}\otimes e_{\nu_1}\otimes e_{\mu_2}\otimes e_{\nu_2})\theta = e_{\mu_1}\otimes e_{\mu_2}\otimes e_{\nu_1}\otimes e_{\nu_2}$.

It also follows easily that

$$(T^{\ell_1}(f_1)\otimes T^{m_1}(f_2)\otimes T^{\ell_2}(f_1)\otimes T^{m_2}(f_2))^{\theta^{-1}} = T^{\ell_1+\ell_2}(f_1)\otimes T^{m_1+m_2}(f_2),$$

where $(a_1\otimes\cdots\otimes a_m)^{\theta^{-1}}$ is defined to be $a_{\theta(1)}\otimes\cdots\otimes a_{\theta(m)}$.

Finally, the desired automorphism ψ is given by $\psi(\underline{a}\otimes\sigma)=\underline{a}^{\theta^{-1}}\otimes\theta^{-1}\sigma\theta$.

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